

# Area (or Entropy) Bound of Horizons for Hořava Lifshitz Black Hole

Parthapratim Pradhan\*

Department of Physics, Vivekananda Satabarshiki Mahavidyalaya, West Midnapur 721513, India

## Abstract

We speculate various thermodynamic features of the inner horizon ( $\mathcal{H}^-$ ) and outer horizon ( $\mathcal{H}^+$ ) in the background of Hořava Lifshitz black hole (BH). We compute particularly *area sum*, *area minus* and *area division* of the BH horizons. We find that they all are *not* showing universal behavior whereas the product is an universal quantity [7]. Based on these relations, we derive the area bound of all horizons. From area bound we derive entropy bound and irreducible mass bound for all the horizons ( $\mathcal{H}^\pm$ ). We also observe that the *First law* of BH thermodynamics and *Smarr-Gibbs-Duhem* relations do not hold for this BH. The underlying reason behind this failure due to the scale invariance of the coupling constant. Moreover, we compute the *Cosmic-Censorship-Inequality* for this BH which gives the lower bound for the total mass of a spacetime and it is supported by cosmic censorship conjecture.

## 1 Introduction

Recently the general relativity community and the string theory community have become quite interested to examine the thermodynamic features of  $\mathcal{H}^-$  and  $\mathcal{H}^+$  [1, 2, 3, 4, 5, 6, 8, 9]. Of particular interest are relations that are independent of mass so called ADM (Arnowitt-Deser-Misner) mass and then these relations are said to be “universal” in BH physics. They are novel in the sense that they involve the thermodynamic quantities defined at multi-horizons, i.e. the Cauchy (inner) horizon and event (outer) horizons of spherically symmetric charged, axisymmetric charged and axisymmetric non-charged black hole. For example, let us consider first spherically symmetric charged BH, i.e. Reissner Nordström (RN) BH, the mass-independent relation for both the horizons ( $\mathcal{H}^\pm$ ) becomes

$$\mathcal{A}_+\mathcal{A}_- = (4\pi Q^2)^2 \text{ or } \mathcal{S}_+\mathcal{S}_- = (\pi Q^2)^2. \quad (1)$$

For spinning non-charged BH, i.e. for Kerr BH, the mass independent relations are

$$\mathcal{A}_+\mathcal{A}_- = (8\pi J)^2 \text{ or } \mathcal{S}_+\mathcal{S}_- = (2\pi J)^2. \quad (2)$$

Finally, for charged spinning BH, these relations should read off

$$\mathcal{A}_+\mathcal{A}_- = (8\pi J)^2 + (4\pi Q^2)^2 \text{ or } \mathcal{S}_+\mathcal{S}_- = (2\pi J)^2 + (\pi Q^2)^2. \quad (3)$$

Remarkably, all these thermodynamic relations are independent of the mass parameter therefore it should be treated as an universal quantity.

For BPS (Bogomol’ni-Prasad-Sommerfield) class of BHs, the area product formula [2] of  $\mathcal{H}^\pm$  should be written as

$$\mathcal{A}_+\mathcal{A}_- = (8\pi)^2 \left( \sqrt{N_L} + \sqrt{N_R} \right) \left( \sqrt{N_L} - \sqrt{N_R} \right) = N, \quad N \in \mathbb{N}, N_1 \in \mathbb{N}, N_2 \in \mathbb{N}. \quad (4)$$

where the integers  $N_L$  and  $N_R$  should be defined as excitation numbers of the left and right moving sectors of a weakly-coupled 2D conformal field theory (CFT). Resultantly, the entropy product formula of  $\mathcal{H}^\pm$  becomes

$$\mathcal{S}_+\mathcal{S}_- = (2\pi)^2 \left( \sqrt{N_L} + \sqrt{N_R} \right) \left( \sqrt{N_L} - \sqrt{N_R} \right) = N, \quad N \in \mathbb{N}, N_1 \in \mathbb{N}, N_2 \in \mathbb{N}. \quad (5)$$

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\*pppradhan77@gmail.com

Therefore it implies that the product of entropy of  $\mathcal{H}^\pm$  is an integer quantity [10].

The product formulae that we will derive in this work, either area (or entropy) product of inner horizon and outer horizons could be used to determine whether the corresponding Bekenstein-Hawking entropy may be written as a Cardy formula, therefore providing some evidences for a CFT description of the corresponding microstates [3, 4, 11]. This boosts to study the properties of the inner horizon thermodynamics in contrast with outer horizon thermodynamics.

In our previous study [7], we derived the surface area product, BH entropy product, surface temperature product, Komar energy product and specific heat product for this BH. Besides the area or entropy product it should be known what happens in case of *area sum*, *area minus* and *area division*. For this reason we extend our study by computing area sum, entropy sum, temperature sum and specific heat sum of all the horizons. We expect that the quantization area product formula that we have found from our previous investigation and from present study both provides a strong indication that there exists an universal near-horizon structure for more general class of BHs. This indicates the possibility that the microscopic degrees of freedom may admit a dual field theoretic explanation that generalizes the 2D CFT duals.

Thus in this Letter, we wish to examine various thermodynamic features (besides the area or entropy product) of Kehagias-Sfetsos BH [13] in Hořava Lifshitz gravity [14, 15, 16]. We have considered both the inner horizon and outer horizons to further understanding the microscopic nature of BH entropy both interior and exterior. Moreover using these relations, we derive the area bound of all horizons. From area bound we derive entropy bound and irreducible mass bound for both the horizons.

One aspect that has not been studied previously which is so called *Cosmic-Censorship-Inequality* or *Cosmic Censorship Bound* [18]. It should require the cosmic-censorship hypothesis [17] (See [19, 20, 21, 22, 23]) and which is an important inequality in general relativity which relates the total mass of a spacetime in terms of the  $\mathcal{H}^+$  area and for Schwarzschild BH it should be minimum i.e.

$$\mathcal{M} \geq \sqrt{\frac{\mathcal{A}_+}{16\pi}}. \quad (6)$$

This brilliant idea had first given Penrose in 1973 [17]<sup>1</sup>.

This inequality has an important implication in BH physics that it indicates the lower bound on the energy for a time-symmetric initial Cauchy data set which satisfied the Einstein equations, and which has also satisfied the dominant energy condition and which has no naked singularities.

The structure of the paper is as follows. In Sec. 2, we shall describe various thermodynamic features of Kehagias-Sfetsos BH in Hořava Lifshitz gravity, we also calculate the different thermodynamic bound in different subsections. Finally, we conclude our discussions in Sec. 3.

## 2 Thermodynamic Properties of Hořava Lifshitz BH:

In 2009, Hořava[14, 15, 16] gave a beautiful field theory model for a UV complete theory of gravity which is a non-relativistic renormalizable theory of gravity and reduces to Einstein's general relativity at large scales for the dynamical coupling constant  $\lambda = 1$ . We have not mentioned the ADM formalism here because it has been already mentioned in[7]. Since we are interested in this work to study the thermodynamic properties of Kehagias-Sfetsos(KS) BH[13] in Hořava Lifshitz(HL) gravity thus the metric of KS BH [26, 27, 13, 28, 29, 30] is given by

$$ds^2 = -\mathcal{F}(r)dt^2 + \frac{dr^2}{\mathcal{F}(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (7)$$

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<sup>1</sup> This beautiful argument can translate into a very interesting mathematical inequality in Riemannian geometry which is so called *Riemannian Penrose Inequality*. It was first examined and proved by G. Huisken et al. [25]. This inequality has an important application in gravitational collapse and using Cauchy data it could be solved the Einsteins equations. Finally, it has another interesting application to solve the Yamabe problem [19]. It should be noted that *Riemannian Penrose Inequality* satisfied the *Riemannian positive mass theorem* [24].

where,

$$\mathcal{F}(r) = 1 - \sqrt{4\mathcal{M}\omega r + \omega^2 r^4} + \omega r^2, \quad (8)$$

and  $\mathcal{M}$  is an integration constant related to the mass parameter. For  $r \gg (\frac{\mathcal{M}}{\omega})^{\frac{1}{3}}$ , we obtain the usual behavior of a Schwarzschild BH.

The BH horizons correspond to  $\mathcal{F}(r) = 0$ :

$$r_{\pm} = \mathcal{M} \pm \sqrt{\mathcal{M}^2 - \frac{1}{2\omega}}. \quad (9)$$

where  $r_+$  is event horizon and  $r_-$  is Cauchy horizon respectively. As long as

$$\mathcal{M}^2 - \frac{1}{2\omega} \geq 0. \quad (10)$$

then the KS metric describes a BH, otherwise it has a naked singularity. When  $\mathcal{M}^2 - \frac{1}{2\omega} = 0$ , we find the extremal KS BH.

The product and sum of horizon radii becomes

$$r_+ r_- = \frac{1}{2\omega} \text{ and } r_+ + r_- = 2\mathcal{M}. \quad (11)$$

The area[7] of this BH is given by

$$\mathcal{A}_{\pm} = 4\pi \left( 2\mathcal{M}r_{\pm} - \frac{1}{2\omega} \right) \quad (12)$$

Their product[7] and sum yields

$$\mathcal{A}_+ \mathcal{A}_- = \frac{4\pi^2}{\omega^2} \text{ and } \mathcal{A}_+ + \mathcal{A}_- = 4\pi \left( 4\mathcal{M}^2 - \frac{1}{\omega} \right). \quad (13)$$

It is remarkable that the area product of KS BH is independent of mass but the area sum is not independent of the mass parameter.

For completeness, we further compute the area minus and area division:

$$\mathcal{A}_{\pm} - \mathcal{A}_{\mp} = \pm 16\pi\mathcal{M}\sqrt{\mathcal{M}^2 - \frac{1}{2\omega}}. \quad (14)$$

and

$$\frac{\mathcal{A}_+}{\mathcal{A}_-} = \frac{r_+^2}{r_-^2}. \quad (15)$$

Again, the sum of area inverse is found to be

$$\frac{1}{\mathcal{A}_+} + \frac{1}{\mathcal{A}_-} = \frac{\omega^2}{\pi} \left( 4\mathcal{M}^2 - \frac{1}{\omega} \right). \quad (16)$$

and the minus of area inverse is computed to be

$$\frac{1}{\mathcal{A}_{\pm}} - \frac{1}{\mathcal{A}_{\mp}} = \mp \frac{4\omega^2\mathcal{M}}{\pi} \sqrt{\mathcal{M}^2 - \frac{1}{2\omega}}. \quad (17)$$

It indicates that they are all mass dependent relations.

Likewise, the entropy product [7] and entropy sum of  $\mathcal{H}^{\pm}$  becomes:

$$\mathcal{S}_- \mathcal{S}_+ = \frac{\pi^2}{4\omega^2} \text{ and } \mathcal{S}_- + \mathcal{S}_+ = \pi \left( 4\mathcal{M}^2 - \frac{1}{\omega} \right). \quad (18)$$

For our record, we also compute the entropy minus of  $\mathcal{H}^\pm$  as

$$\mathcal{S}_\pm - \mathcal{S}_\mp = \pm 4\pi\mathcal{M}\sqrt{\mathcal{M}^2 - \frac{1}{2\omega}}. \quad (19)$$

and the entropy division of  $\mathcal{H}^\pm$  as

$$\frac{\mathcal{S}_+}{\mathcal{S}_-} = \frac{r_+^2}{r_-^2}. \quad (20)$$

Again, the sum of entropy inverse is found to be

$$\frac{1}{\mathcal{S}_+} + \frac{1}{\mathcal{S}_-} = \frac{4\omega^2}{\pi} \left( 4\mathcal{M}^2 - \frac{1}{\omega} \right). \quad (21)$$

and the minus of entropy inverse is

$$\frac{1}{\mathcal{S}_\pm} - \frac{1}{\mathcal{S}_\mp} = \mp \frac{16\omega^2\mathcal{M}}{\pi} \sqrt{\mathcal{M}^2 - \frac{1}{2\omega}}. \quad (22)$$

The Hawking [34] temperature on  $\mathcal{H}^\pm$  reads off

$$T_\pm = \frac{\omega(r_\pm - \mathcal{M})}{2\pi(1 + \omega r_\pm^2)}. \quad (23)$$

Their product [7] and sum yields

$$T_+T_- = \frac{\omega(1 - 2\mathcal{M}^2\omega)}{2\pi^2(1 + 16\mathcal{M}^2\omega)} \text{ and } T_+ + T_- = \frac{4\omega\mathcal{M}(1 - 2\mathcal{M}^2\omega)}{\pi(1 + 16\mathcal{M}^2\omega)}. \quad (24)$$

It may be noted that surface temperature product and sum both depends on mass thus they are not universal in nature. It is shown that for KS BH:

$$T_+\mathcal{S}_+ + T_-\mathcal{S}_- = \frac{8\omega\mathcal{M}\sqrt{\mathcal{M}^2 - \frac{1}{2\omega}}}{1 + 16\omega\mathcal{M}^2}. \quad (25)$$

In general, this relation is for RN BH or Kerr BH [38]:

$$T_+\mathcal{S}_+ + T_-\mathcal{S}_- = 0. \quad (26)$$

It is in-fact a mass independent (universal) relation and implies that  $T_+\mathcal{S}_+ = T_-\mathcal{S}_-$  should be taken as a criterion whether there is a 2D CFT dual for the BHs in the Einstein gravity and other diffeomorphism gravity theories [11, 32]. This universal relation also indicates that the left and right central charges are equal i.e.  $c_L = c_R = 12J$ , which is holographically dual to 2D CFT [37].

But for KS BH, we see that it is mass dependent. It is also interesting to mention that except the area product, entropy product and irreducible mass product all the thermodynamic relations of KS BH are mass dependent.

## 2.1 Smarr Formula for HL BH on $\mathcal{H}^\pm$ :

Smarr [33] had first derived the ADM mass can be expressed as in terms of area, angular momentum and charge for Kerr-Newman black hole. On the otherway, Hawking [34] has been speculated that the BH area always increases. Therefore the BH area is indeed a constant quantity over the  $\mathcal{H}^\pm$ . Analogously, the area of both the horizons for KS BH in HL gravity is given by

$$\mathcal{A}_\pm = 4\pi \left[ 2\mathcal{M}^2 - \frac{1}{2\omega} \pm 2\mathcal{M}\sqrt{\mathcal{M}^2 - \frac{1}{2\omega}} \right] \quad (27)$$

Alternatively, the mass could be expressed as in terms of horizons ( $\mathcal{H}^\pm$ ):

$$\mathcal{M}^2 = \frac{\mathcal{A}_\pm}{16\pi} + \frac{\pi}{4\omega^2\mathcal{A}_\pm} + \frac{1}{4\omega}. \quad (28)$$

Form the above relation we can easily derived the *Cosmic-Censorship-Inequality* for KS BH:

$$\mathcal{M} \geq \sqrt{\frac{\mathcal{A}_\pm}{16\pi} + \frac{\pi}{4\omega^2\mathcal{A}_\pm} + \frac{1}{4\omega}}. \quad (29)$$

Actually, Penrose derived it for  $\mathcal{H}^+$  only. We here suggests this inequality is valid for  $\mathcal{H}^-$  also.

After differentiation, we get the mass differential as

$$d\mathcal{M} = \mathcal{T}_\pm d\mathcal{A}_\pm + \Phi_\pm^\omega d\omega \quad (30)$$

where,

$$\begin{aligned} \mathcal{T}_\pm &= \text{Effective surface tension for horizons} \\ &= \frac{1}{\mathcal{M}} \left( \frac{1}{32\pi} - \frac{\pi}{8\omega^2\mathcal{A}_\pm^2} \right) = \frac{\partial\mathcal{M}}{\partial\mathcal{A}_\pm}. \end{aligned} \quad (31)$$

$$\begin{aligned} \Phi_\pm^\omega &= \text{Effective potential for horizons due to } \omega \\ &= -\frac{1}{\mathcal{M}} \left( \frac{\pi}{4\omega^3\mathcal{A}_\pm} + \frac{1}{8\omega^2} \right) = \frac{\partial\mathcal{M}}{\partial\omega}. \end{aligned} \quad (32)$$

It is well known that for Spherically symmetric RN BH, the Smarr-Gibbs-Duhem relation is satisfied by the following condition:

$$\frac{\mathcal{M}}{2} - T_\pm \mathcal{S}_\pm - \frac{\Phi_\pm}{2} Q = 0. \quad (33)$$

where the symbols are used as usual for RN BH. But for KS BH this relation is

$$\frac{\mathcal{M}}{2} - T_\pm \mathcal{S}_\pm - \frac{\Phi_\pm^\omega}{2} \omega = \frac{2 + 5\omega r_\pm^2}{8\omega r_\pm(1 + \omega r_\pm^2)} \neq 0. \quad (34)$$

It indicates, the Smarr-Gibbs-Duhem relation do not satisfied for KS BH in HL gravity. Followed by the first law of BH thermodynamics also do not satisfied for this BH. The reason should be due to the scale invariance of the coupling constant  $\omega$ . This observation is essential here because we have not seen such type of discussion in the literature regarding the KS BH in HL gravity.

It should be emphasized that when we add the AdS term to this BH then the both first law of thermodynamics and Smarr-Gibbs-Duhem relations have satisfied which has been explicitly examined in [36]. Where the author derived the generalized Smarr relation in AdS space which has include a pressure-volume term and the thermodynamic mass, ADM mass, Brown-York mass and Holland-Ishibashi-Marolf mass could also be defined. But it is interesting to note that with out pressure-volume term the first law and Smarr relation do not satisfied at all. This is one of the key result of our work.

## 2.2 Area Bound of KS BH for $\mathcal{H}^\pm$ :

Using the above thermodynamic relations, we are now able to derive the entropy bound of both the horizons. Using the inequality Eq. (10) one can obtain  $\mathcal{M}^2 \geq \frac{1}{2\omega}$ . Since  $r_+ \geq r_-$ , one can get  $\mathcal{A}_+ \geq \mathcal{A}_- \geq 0$ . Then the area product gives:

$$\mathcal{A}_+ \geq \sqrt{\mathcal{A}_+\mathcal{A}_-} = \frac{2\pi}{\omega} \geq \mathcal{A}_-. \quad (35)$$

and the area sum gives:

$$4\pi \left(4\mathcal{M}^2 - \frac{1}{\omega}\right) = \mathcal{A}_+ + \mathcal{A}_- \geq \mathcal{A}_+ \geq \frac{\mathcal{A}_+ + \mathcal{A}_-}{2} = 2\pi \left(4\mathcal{M}^2 - \frac{1}{\omega}\right). \quad (36)$$

Thus the area bound for  $\mathcal{H}^+$ :

$$2\pi \left(4\mathcal{M}^2 - \frac{1}{\omega}\right) \leq \mathcal{A}_+ \leq 4\pi \left(4\mathcal{M}^2 - \frac{1}{\omega}\right). \quad (37)$$

and the area bound for  $\mathcal{H}^-$ :

$$0 \leq \mathcal{A}_- \leq \frac{2\pi}{\omega}. \quad (38)$$

### 2.3 Entropy Bound for $\mathcal{H}^\pm$ :

Analogously, as  $r_+ \geq r_-$ , one can get  $\mathcal{S}_+ \geq \mathcal{S}_- \geq 0$ . Then the entropy product gives:

$$\mathcal{S}_+ \geq \sqrt{\mathcal{S}_+ \mathcal{S}_-} = \frac{\pi}{2\omega} \geq \mathcal{S}_-. \quad (39)$$

and the entropy sum gives:

$$\pi \left(4\mathcal{M}^2 - \frac{1}{\omega}\right) = \mathcal{S}_+ + \mathcal{S}_- \geq \mathcal{S}_+ \geq \frac{\mathcal{S}_+ + \mathcal{S}_-}{2} = \pi \left(2\mathcal{M}^2 - \frac{1}{2\omega}\right). \quad (40)$$

Thus the entropy bound for  $\mathcal{H}^+$ :

$$\pi \left(2\mathcal{M}^2 - \frac{1}{2\omega}\right) \leq \mathcal{S}_+ \leq \pi \left(4\mathcal{M}^2 - \frac{1}{\omega}\right). \quad (41)$$

and the area bound for  $\mathcal{H}^-$ :

$$0 \leq \mathcal{S}_- \leq \frac{\pi}{2\omega}. \quad (42)$$

### 2.4 Irreducible Mass Bound for $\mathcal{H}^\pm$ :

Christodoulou[35] had given a relation between surface area of the  $\mathcal{H}^+$  and irreducible mass, which can be written as

$$\mathcal{M}_{\text{irr},+}^2 = \frac{\mathcal{A}_+}{16\pi} = \frac{\mathcal{S}_+}{4\pi}. \quad (43)$$

It is now well known that this relation is valid for CH also. That means

$$\mathcal{M}_{\text{irr},-}^2 = \frac{\mathcal{A}_-}{16\pi} = \frac{\mathcal{S}_-}{4\pi}. \quad (44)$$

Now the the product and sum of the irreducible mass for both the horizons are

$$\mathcal{M}_{\text{irr},+} \mathcal{M}_{\text{irr},-} = \frac{1}{8\omega} \text{ and } \mathcal{M}_{\text{irr},+}^2 + \mathcal{M}_{\text{irr},-}^2 = \mathcal{M}^2 - \frac{1}{4\omega}. \quad (45)$$

From the area bound, we get the irreducible mass bound for KS BH.

For  $\mathcal{H}^+$ :

$$\frac{\sqrt{4\mathcal{M}^2 - \frac{1}{\omega}}}{2\sqrt{2}} \leq \mathcal{M}_{\text{irr},+} \leq \frac{\sqrt{4\mathcal{M}^2 - \frac{1}{\omega}}}{2}. \quad (46)$$

and for  $\mathcal{H}^-$ :

$$0 \leq \mathcal{M}_{\text{irr},-} \leq \sqrt{\frac{1}{8\omega}}. \quad (47)$$

Eq. 46 is nothing but the Penrose inequality, which is the first geometric inequality for BHs[?].

## 2.5 Temperature Bound for $\mathcal{H}^\pm$ :

In BH thermodynamics, temperature is an important parameter. So there must exist temperature bound relation on the horizons. As is when  $r_+ \geq r_-$ , one must obtain  $T_+ \geq T_- \geq 0$ . Then the temperature product gives:

$$T_+ \geq \sqrt{T_+ T_-} = \sqrt{\frac{\omega(1-2\mathcal{M}^2\omega)}{2\pi^2(1+16\mathcal{M}^2\omega)}} \geq T_- . \quad (48)$$

and the temperature sum gives:

$$\frac{4\omega\mathcal{M}(1-2\mathcal{M}^2\omega)}{\pi(1+16\mathcal{M}^2\omega)} = T_+ + T_- \geq T_+ \geq \frac{T_+ + T_-}{2} = \frac{2\omega\mathcal{M}(1-2\mathcal{M}^2\omega)}{\pi(1+16\mathcal{M}^2\omega)} . \quad (49)$$

Thus the temperature bound for  $\mathcal{H}^+$ :

$$\frac{2\omega\mathcal{M}(1-2\mathcal{M}^2\omega)}{\pi(1+16\mathcal{M}^2\omega)} \leq T_+ \leq \frac{4\omega\mathcal{M}(1-2\mathcal{M}^2\omega)}{\pi(1+16\mathcal{M}^2\omega)} . \quad (50)$$

and the temperature bound for  $\mathcal{H}^-$ :

$$0 \leq T_- \leq \sqrt{\frac{\omega(1-2\mathcal{M}^2\omega)}{2\pi^2(1+16\mathcal{M}^2\omega)}} . \quad (51)$$

## 2.6 Bound on Heat Capacity $C_\pm$ for $\mathcal{H}^\pm$ :

In BH thermodynamics, the specific heat can be defined as

$$C_\pm = \frac{\partial \mathcal{M}}{\partial T_\pm} . \quad (52)$$

which is an important parameter to determine the thermodynamic properties in BH physics. In our previous work[7], we derived in detail the expression for specific heat for both the horizons. It is given by

$$C_\pm = \frac{2\pi}{\omega} \frac{(2\omega r_\pm^2 - 1)(1 + \omega r_\pm^2)^2}{1 + 5\omega r_\pm^2 - 2\omega^2 r_\pm^4} . \quad (53)$$

Their product [7] and sum on  $\mathcal{H}^\pm$  yields:

$$C_+ C_- = \frac{\pi^2}{2\omega^2} \frac{(1-2\mathcal{M}^2\omega)(1+16\mathcal{M}^2\omega)^2}{(2+13\omega\mathcal{M}^2-16\omega^2\mathcal{M}^4)} . \quad (54)$$

and

$$C_+ + C_- = \frac{\pi}{\omega^2} \frac{(128\omega^4\mathcal{M}^6 + 8\omega^3\mathcal{M}^4 - 42\omega^2\mathcal{M}^2 + 4\omega\mathcal{M}^2 + 2\omega - 1)}{(2+13\omega\mathcal{M}^2-16\omega^2\mathcal{M}^4)} . \quad (55)$$

Using  $\mathcal{M}^2 \geq \frac{1}{2\omega}$  with the product of heat capacity and the sum of heat capacity, we get the bound on heat capacity for both the horizons. For  $\mathcal{H}^+$ :

$$\begin{aligned} & \frac{\pi}{2\omega^2} \frac{(128\omega^4\mathcal{M}^6 + 8\omega^3\mathcal{M}^4 - 42\omega^2\mathcal{M}^2 + 4\omega\mathcal{M}^2 + 2\omega - 1)}{(2+13\omega\mathcal{M}^2-16\omega^2\mathcal{M}^4)} \\ & \leq C_+ \leq \end{aligned}$$

$$\frac{\pi}{\omega^2} \frac{(128\omega^4 \mathcal{M}^6 + 8\omega^3 \mathcal{M}^4 - 42\omega^2 \mathcal{M}^2 + 4\omega \mathcal{M}^2 + 2\omega - 1)}{(2 + 13\omega \mathcal{M}^2 - 16\omega^2 \mathcal{M}^4)}. \quad (56)$$

and for  $\mathcal{H}^-$ :

$$0 \leq C_- \leq \sqrt{\frac{\pi}{\omega^2} \frac{(128\omega^4 \mathcal{M}^6 + 8\omega^3 \mathcal{M}^4 - 42\omega^2 \mathcal{M}^2 + 4\omega \mathcal{M}^2 + 2\omega - 1)}{(2 + 13\omega \mathcal{M}^2 - 16\omega^2 \mathcal{M}^4)}}. \quad (57)$$

It should be mentioned that all the above thermodynamic formulae might be suggested the possibility of an explanation for the microscopic nature of such BHs in terms of a field theory in more than two dimensions.

### 3 Discussion:

In order to understand the BH entropy (both outer as well as inner) at the microscopic level, we studied thermodynamic properties of KS BH in HL gravity. We computed various thermodynamic formula for this BH. We speculated that area sum, area minus and area division are mass dependent quantities, whereas the product [7] is a mass independent quantity.

Based on these relations, we computed area bound, entropy bound, irreducible mass bound, temperature bound and specific heat bound. The upper area bound of outer horizon is actually the Penrose-like inequality in BH mechanics. Due to the scale invariance of the coupling constant parameter  $\omega$ , we showed that the First law of black hole thermodynamics and Smarr-Gibbs-Duhem relations do not satisfied for this BH. Finally, we derived the Cosmic-Censorship-Inequality for this BH which has an important implications in Cosmic-Censorship-Conjecture.

In conclusion, these thermodynamic product formulae suggests further evidence for the crucial role of both inner horizon and outer horizon for understanding the microscopic nature of BH entropy (both interior and exterior) which is the prime aim in quantum gravity.

### Acknowledgements

The author is grateful to the authority of Inter-University Centre for Astronomy and Astrophysics(IUCAA), Pune for warm hospitality during a ‘‘Refresher course in Astronomy and Astrophysics’’ and where the most of the work was done. I also would like to thank Dr. W. G. Brenna for useful correspondence.

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